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Stress singularity at a crack tip for various intermediate zones in bimaterial structures (mode III)

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Abstract

The influence of the geometry of a thin intermediate zone on the stress distribution has been investigated in the vicinity of a crack tip in a bimaterial structure. Corresponding modelling boundary value problems are reduced to functional-difference equations by the Mellin transform technique, and later to singular integral equations with fixed point singularities. It has been observed that the order of the stress singularity is essentially dependent on the model parameters. Numerical results concerning the stress singularity exponents and generalized stress intensity factors are presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

In this paper we shall discuss the problems of the stress singularities at the crack tip in the solids with step-wise nonhomogeneities. The usual, traditional method of dealing with such problems consists in finding the solution for a bimaterial solid with ideal bonding between the interfacial boundary, i.e. when it can be assumed that the respective stress tensor components, as well as the displacement vector components vary continuously across the interface. In these cases the boundary conditions along the interface are called "ideal". It is known from the literature on the subject that the solutions with the "ideal" bonding conditions have at least two disadvantages.

Beginning from the papers by Zak and Williams (1963), it is evident that for a crack perpendicular to the interface the asymptotics of the displacements takes the form $u \sim K_m r^{\omega_m} f_m(\theta)$ for $r \to 0$ where the real number $\omega_m \in (0, 1)$ depends on the elastic parameters of the material, the state of stresses $(m = 1, 2, 3)$ and the form of the boundary condition over the crack faces. However when a crack is no longer perpendicular to the bonding plane or in the case of an interfacial crack, parameters ω_m $(m = 1, 2)$, in general, are complex numbers. This fact results in unacceptable, from the physical point of view, oscillatory stresses in the neighbourhood of the crack tip and overlapping crack faces (Comninou, 1979; Rice, 1988). It is interesting to note that even in the case when a crack is perpendicular to the boundary of the "ideal" contact, the stress oscillation at the vicinity of the crack tip can also appear. This may happen when the mixed boundary conditions of the type

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($\sigma_\theta = -p, u_r = u$) or ($u_\theta = u, \sigma_{r\theta} = -p$) are given along the crack surfaces, and at the same time the material constants of elasticity μ_j and $\kappa_j = 3-4v_j$ (where μ_j are the shear moduli while v_j Poisson's ratios, respectively), satisfying the relationships: $(\mu_1-\mu_0)(\mu_1\kappa_0-\mu_0\kappa_1) < 0$ (compare Mishuris, 1986).

Though the use of the so-called "kinked crack approach" (see He and Hutchinson, 1989a, b) makes it possible to avoid the first of the above mentioned disadvantages (the oscillation), nevertheless the model of the "ideal" contact bonding is not capable to take into account any effect of the mechanical characteristics of the contact region on the distribution of stresses or displacement fields. This is the second shortage of the "ideal interface concept". Furthermore "the kinked crack approach" leads to the certain type of "non-uniqueness" with regards to the crack extension process.

On the other hand, if we take into account the existence of a thin intermediate zone we shall be able to consider the effect of the contacting regions. As examples we can cite the papers by Atkinson (1977), Erdogan et al. (1991) and Mishuris (1985). If we assume that the intermediate elastic zone is of constant thickness h_* and its mechanical properties are such that their values change continuously from those of the first material to the values for the second material, we shall obtain the value of parameter ω_m equal to 0.5, and the distribution of stresses at a crack tip resembles that encountered for a homogeneous elastic solid. However, in the case when the thickness of the intermediate zone is essentially smaller than a crack length l_* and the characteristic dimension D_*, stress intensity factor K_m will assume either very small or very large values depending on the ratios of the elastic parameters characterizing the materials (Atkinson and Javaherian, 1980; Erdogan et al., 1991). A similar situation arises in the case of the "kinked crack approach" when the length of the kinked crack tends to zero.

In fact the surfaces of the two bonded materials as a rule are not identical. There exist various asperities and roughnesses of the bonded surface. If the height of the asperities is much smaller than the thickness of the intermediate zone then the fact that the surfaces of the bonded materials are not smooth does not play any essential role. However, when the above parameters are comparable in value, the situation can be entirely different.

The purpose of the paper is to investigate the behaviour of stress tensor and displacement vector in the neighbourhood of a crack tip in the case when the crack terminates at a thin intermediate zone for various kinds of geometry. It is assumed that the characteristic thickness of the intermediate zone satisfies the relations: $h_* \ll l_*$, $h_* \ll D$ (see Fig. 1).

In order to discuss a sufficiently general case we assume that the thickness of the elastic intermediate zone (inclusion) can be written down in the form:

$$h(x_*) = h_* \cdot |x_*/d_*|^\alpha, \quad |x_*| \leqslant d_*, \quad 0 \leqslant \alpha < \infty, \tag{1}$$

where x_* denotes the distance from a crack tip to a point on the contact bonding surface of two materials, and $d_* \ll D_*$ denotes a certain constant which can be interpreted as a characteristic linear dimension of asperities and roughnesses on the surfaces of the bonded materials.

Thus, making use of the standard assumptions, applicable for the problems of thin inclusions, we obtain the following contact conditions along the mid surface Γ of the pertinent segment of the intermediate zone:

$$[\sigma_n]_{|\Gamma} = 0, \quad ([u]-\tau^*\sigma_n)_{|\Gamma} = 0, \tag{2}$$

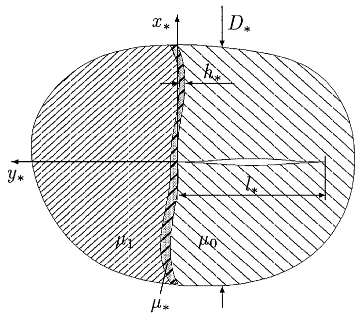

Fig. 1. Bimaterial solid with a crack.

where $[u]$, $[\sigma_n]$ denote the jumps of the displacement and stress vectors between two points of material 1 and material 0 separated by the thin intermediate zone. Besides the diagonal matrix τ^* have the parameters determined from the formulae:

$$\tau_1^* = h(x_*)/\mu_{xy}^*, \quad \tau_2^* = h(x_*)/E^*, \quad \tau_3^* = h(x_*)/\mu_{yz}^*, \tag{3}$$

where E^* and μ_{xy}^*, μ_{yz}^* denote Young's modulus, and the shear strain modulus of an anisotropic inclusion.

Let us observe that the relations (2) and (3) have been obtained by Cherepanov (1979) in a little bit different context. Similar contact conditions (at least for $\alpha = 0$) appear frequently in the rocks mechanics. Then the coefficients of matrix τ^* are determined from experiments.

Now, we can norm all the quantities possessing the dimension of length by means of the characteristic length d_*. In this way we arrive at a model problem of finding the dimensionless displacements in an infinite bimaterial plane, weakened by a semi-infinite crack. On the surface of the bonding of two materials $y(=y_*/d_*) = 0$ the following bonding conditions have to be satisfied:

$$[\sigma_n]\,|_{y=0} = 0, \quad ([u] - \boldsymbol{\tau} r^{\alpha}\sigma_n)\,|_{y=0} = 0, \tag{4}$$

where the non-negative components of the diagonal matrix $\boldsymbol{\tau}$ are constant and can be determined either experimentally, or from the formulae resulting from (3):

$$\tau_1 = \frac{h_*}{d_*\mu_{xy}^*}, \quad \tau_2 = \frac{h_*}{d_* E^*}, \quad \tau_3 = \frac{h_*}{d_*\mu_{yz}^*}. \tag{5}$$

The corresponding forms of the intermediate zone near the crack tip have been shown in Fig. 2b–e. Let us note that an ''ideal contact'' can be obtained from (4), as a special case, by assuming that $\boldsymbol{\tau} = 0$ (Fig. 2a).

Here, in fact, we have made use of the assumption that $h_* \ll d_* \ll D_*$. In the case when the quantities h_* and d_* are of the same order of smallness, then the surface of material contact Γ, as a rule is no longer a plane $y = 0$.

We confine our consideration to the simplest case, i.e. of the antiplane stress. Cases $\alpha = 0$ and $\alpha = 1$ have been already discussed in Mishuris (1997).

In the second and third sections the problem is formulated and reduced to a functional-difference equation. In the next section, the solution of the equation is found for any of the values of the

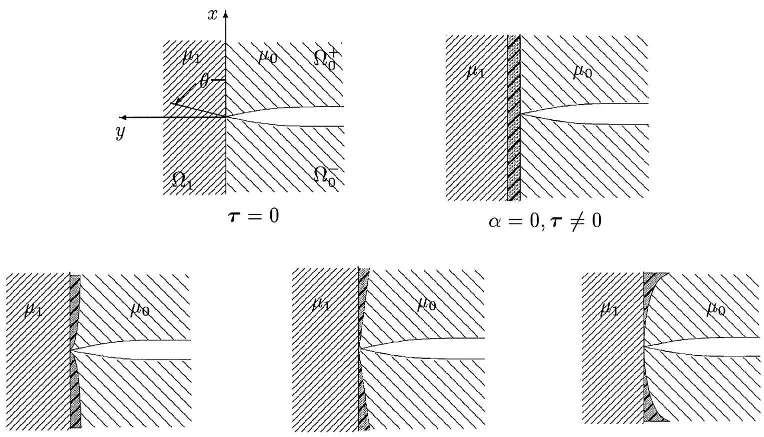

Fig. 2. Different geometry of the thin modelling interfacial zone near the crack tip.

parameters based on the results in the Appendix. In the fifth section, asymptotics of the displacement field near the crack tip is obtained and the singularity exponent of stresses is calculated. In the last two sections numerical results are presented and discussed.

2. Problem formulation

Let us consider the modelling problem for a bimaterial plane with a semi-infinity crack terminating perpendicularly at the interface. We shall seek for displacements $u_1(x,y)$, $u_0^{\pm}(r,\theta)$ in the domains Ω_1, $\Omega_0^{\pm}$ (see Fig. 2a):

$$\Omega_1 = \{(x,y)\colon -\infty < x < \infty, 0 < y < \infty\},$$

$$\Omega_0^{\pm} = \{(r,\theta)\colon 0 < r < \infty, \pm(\theta+\pi/2) \in (0,\pi/2)\},$$

which are harmonic functions in the corresponding domains and satisfy exterior boundary conditions along the crack surfaces:

$$\sigma_{\theta z}^{\pm}|_{\Gamma_0^{\pm}} \equiv \mu_0 \frac{1}{r}\frac{\partial}{\partial\theta} u_0^{\pm}|_{\Gamma_0^{\pm}} = -g(r), \tag{6}$$

where $\Gamma_0^{\pm} = \{(r,\theta)\colon 0 < r < \infty, \theta = -\pi/2 \pm 0\}$. We assume throughout the paper that the function $g(r)$ is sufficiently smooth and has a compact support [for example $g \in C_0^{\infty}(0,\infty)$] then singularities of the solutions are connected with the interior properties of the problems. Due to the symmetry of the geometry and conditions (6), we can state that $u_0^-(r,\theta) = u_0^+(r,-\theta-\pi)$ and then the following condition is satisfied:

$$u_1|_{\Gamma_1} = 0, \tag{7}$$

on the crack line ahead $\Gamma_1 = \{(r,\theta)\colon 0 < r < \infty, \theta = \pi/2\}$. Further on, we omit the superscript "+" in the symbol u_0^+.

Along the interior boundary $\Gamma_+ = \{(x, y): 0 < x < \infty, y = 0\}$ between half-plane Ω_1 and wedge Ω_0^+ the interfacial conditions hold true:

$$\left(u_1 - u_0 - \mu_0 \tau r^{\alpha-1} \frac{\partial}{\partial \theta} u_0\right)\Big|_{\Gamma_+} = 0,$$

$$\frac{\partial}{\partial y}(\mu_1 u_1 - \mu_0 u_0)|_{\Gamma_+} = 0, \tag{8}$$

with certain known constants τ, $\alpha \geqslant 0$.

We shall seek for regular solutions $u_1 \in C^2(\Omega_1)$, $u_0 \in C^2(\Omega_0^+)$ of problems (6)–(8) meeting the conditions at the singular points:

$$1^0. \quad u_1, u_0 = \begin{cases} O(r^{\vartheta_0}), & r \to 0, \\ O(r^{-\vartheta_\infty}), & r \to \infty, \end{cases} \tag{9}$$

$$2^0. \quad \nabla u_1, \nabla u_0 = \begin{cases} O(r^{\gamma_0 - 1}), & r \to 0, \\ O(r^{-\gamma_\infty - 1}), & r \to \infty. \end{cases} \tag{10}$$

Here $\vartheta_0, \vartheta_\infty \geqslant 0, \gamma_0, \gamma_\infty > 0$ $(\vartheta_0 + \vartheta_\infty > 0)$ are some unknown constants, which depend on the values of μ_0, μ_1, τ, α and will be calculated from the solution of the problem.

Remark. All variables x, y, r and the sought functions $u_0(x, y)$, $u_1(x, y)$ are dimensionless, the constants μ_0, μ_1, $1/\tau$ and the function $g(r)$ have dimension [N/m^2]. Consequently, constants in the main singular terms of stresses near the crack tip (generalized SIF) have the same dimension.

3. Reduction to a system of functional equations

By applying the Mellin transform in the respective domains to Laplace equation, we obtain

$$\tilde{u}_j(s, \theta) = \int_0^\infty u_j(r, \theta) r^{s-1}\, \mathrm{d}r = A_j(s)\cos(s\theta) + B_j(s)\sin(s\theta), \quad j = 0, 1. \tag{11}$$

From *a priori* estimates (9) and (10) of the solution, and the properties of the Mellin transform, it follows that functions $A_j(s)$, $B_j(s)$ are analytic in the strip $-\min\{\vartheta_0, \gamma_0\} < \Re s < \min\{\vartheta_\infty, \gamma_\infty\}$. Besides, $A_j(s)$ can have a double pole at point $s = 0$, in general, but functions $B_j(s)$ can have only a simple pole at zero. If this occurs, the value of the parameter ϑ_0 is equal to 0. Nevertheless, functions $s^2 A_j(s)$, $sB_j(s)$ are analytic in the strip $-\gamma_0 < \Re s < \gamma_\infty$.

Substituting (11) in (6)–(8)$_\mathrm{b}$ we obtain a system of linear equations in the strip $-\gamma_0 < \Re s < \gamma_\infty$:

$$s^2[A_1(s)\cos(\pi s/2) + B_1(s)\sin(\pi s/2)] = 0,$$

$$\mu_0 s[A_0(s)\sin(\pi s/2) + B_0(s)\cos(\pi s/2)] = -\tilde{g}(s+1),$$

$$\mu_0 s B_0(s) = \mu_1 s B_1(s). \tag{12}$$

Besides, the balance condition for the wedge Ω_0^+ can be additionally written in the form:

$$\mu_0 \lim_{s\to 0} sB_0(s) = -\tilde{g}(1). \tag{13}$$

Consequently, $B_0(s)$, $B_1(s)$ have a simple pole at point $s = 0$; function $A_1(s)$ is analytic in the strip $-\min\{1, \gamma_0\} < \Re s < \min\{1, \gamma_\infty\}$, at least, but from (13) and the second eqn (12), it follows that function $A_0(s)$ can have only a simple pole at point $s = 0$.

The remaining boundary condition $(8)_a$ can be written in the form:

$$A_1(s) - A_0(s) - \mu_0 \tau(s+\alpha-1) B_0(s+\alpha-1) = 0. \tag{14}$$

This functional equation holds true in a strip depending on the value of α. Namely, in the case where $\alpha = 1$, eqn (14) is satisfied in the whole strip $-\gamma_0 < \Re s < \gamma_\infty$, and function $A_0(s)$ has no pole at zero. As a particular case, the corresponding solution of the problem can be obtained from a paper by Mishuris (1997). If $\alpha \in [0, 1)$, then we should seek for solution of eqn (14) in the strip $-\gamma_0 + 1 - \alpha < \Re s < \gamma_\infty$. It means that eqn (14) holds true in a certain strip of the complex argument s with positive real part. In the case $\alpha > 1$, eqn (14) is satisfied in the strip $-\gamma_0 < \Re s < \alpha - 1 + \gamma_\infty$ (for negative real part of the parameter s). Besides, functions $A_0(s)$, $B_0(s+\alpha-1)$ have simple poles at point $s = 0$. It is evident that we should assume

$$\gamma_0 + \alpha + \gamma_\infty > 1. \tag{15}$$

Taking into account (13) we introduce new dimensionless symbol $D(s)$:

$$D(s) = sB_1(s), \quad D(0) = -\mu_1^{-1} \tilde{g}(1), \tag{16}$$

and eqn (14) can be rewritten in terms of function $D(s)$:

$$\frac{1}{s} \Delta(s) D(s) + \frac{\tilde{g}(s+1)}{\mu_1 s \sin(\pi s/2)} = \mu_1 \tau D(s+\alpha-1), \tag{17}$$

where

$$\Delta(s) = \mu_1/\mu_0 \operatorname{ctg}(\pi s/2) - \operatorname{tg}(\pi s/2) = \frac{2(\cos \pi s - \kappa)}{(1+\kappa)\sin \pi s}, \quad \kappa = \frac{\mu_0 - \mu_1}{\mu_0 + \mu_1}.$$

Let ω be the zero of the positive real part of function $\Delta(s)$ nearest to the imaginary axis. One can easily see that ω is real and $\omega \in (0, 1)$. Let us note here that the value of $\omega - 1$ is the stress singularity exponent in the case of the "ideal" bimaterial interface ($\tau = 0$).

4. Solution of the problem

In the case where $\alpha \in [0, 1)$ eqn (17) is valid in the strip $0 < \Re s < \omega$, at least. Hence, function $D(s)$ is analytic in the strip $\alpha - 1 < \Re s < \omega$, and

$$\gamma_0 = 1 - \alpha, \quad \gamma_\infty = \omega, \quad \vartheta_0 = 0, \quad \vartheta_\infty = \omega. \tag{18}$$

At points $s = \alpha - 1$ and $s = \omega$, function $D(s)$ has simple poles, whence

$$a_0^- = \lim_{s\to\alpha-1} (s+1-\alpha) D(s) = \frac{2}{\pi \mu_0 \mu_1 \tau} \{\mu_1 D'(0) + \tilde{g}'(1)\},$$

$$b_0^- = \lim_{s\to\omega}(s-\omega)D(s) = \frac{\omega\mu_0}{\pi(\mu_0+\mu_1)}\left\{\frac{\tilde{g}(1+\omega)}{\mu_0\omega\sin(\pi\omega/2)} - \mu_1\tau D(\alpha-1+\omega)\right\}. \tag{19}$$

It is proved in the Appendix that functional eqn (17) has a unique solution, which can be found from the singular integral eqn (A5), by use of relations (A1) and (A2). Therefore, constant $D'(0)$, $D(\alpha-1+\omega)$ in (19) are calculated by means of solution $f(t)$ of the singular integral eqn (A5):

$$D'(0) = \frac{\pi}{2(1-\alpha)}\tilde{f}(1-\alpha) - \frac{1}{\mu_1(1-\alpha)}\Gamma'(1)\tilde{g}(1),$$

$$D(\alpha-1+\omega) = \Gamma\left(\frac{\omega}{1-\alpha}\right)\cos\frac{\pi\omega}{2(1-\alpha)}\left[\frac{2(1-\alpha)}{\mu_1\pi(\alpha-1+\omega)}\tilde{g}(1) - \tilde{f}(\omega)\right]. \tag{20}$$

Here $\Gamma(s)$ denotes the gamma-function, and $-\Gamma'(1) = \gamma$ is the Euler constant.

Now, consider the case where $\alpha > 1$. By the reasoning similar to that above, one can conclude that eqn (17) holds true in the strip $-\omega < \Re s < 0$. Hence, function $D(s)$ is analytic in the strip $-\omega < \Re s < \alpha-1$, and it has simple poles at points $s = -\omega$ and $s = \alpha-1$:

$$\gamma_0 = \omega, \quad \gamma_\infty = \alpha-1, \quad \vartheta_0 = \omega, \quad \vartheta_\infty = 0. \tag{21}$$

Besides, $b_0^+ = a_0^-$ [see $(19)_a$, $(20)_a$], but the remaining constant is of the form:

$$a_0^+ = \lim_{s\to-\omega}(s+\omega)D(s) = \frac{\omega\mu_0}{\pi(\mu_0+\mu_1)}\left\{\mu_1\tau D(\alpha-1-\omega) - \frac{\tilde{g}(1-\omega)}{\mu_0\omega\sin(\pi\omega/2)}\right\},$$

$$D(\alpha-1-\omega) = \Gamma\left(\frac{\omega}{\alpha-1}\right)\cos\frac{\pi\omega}{2(\alpha-1)}\left[\frac{2(1-\alpha)}{\mu_1\pi(\alpha-1-\omega)}\tilde{g}(1) - \tilde{f}(-\omega)\right]. \tag{22}$$

Let us remember that constants γ_0, γ_∞, ϑ_0, ϑ_∞ in *a priori* estimations (10) have been calculated in (18) and (21), so that assumption (15) is justified for both the cases $0 < \alpha < 1$ and $\alpha > 1$. Besides, note that all constants obtained in this section ($a_0^+, a_0^-, b_0^+, b_0^-$) are dimensionless.

5. Analysis of the solutions

Now we investigate asymptotics of the displacements $u_0(r,\theta)$, $u_1(r,\theta)$ near the crack tip in the respective domains Ω_0, Ω_1. Note that:

$$u_1(r,\theta) = -\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\frac{\sin(\pi/2-\theta)s}{s\cos(\pi s/2)}D(s)r^{-s}\,\mathrm{d}s, \quad \theta\in(0,\pi/2),$$

$$u_0(r,\theta) = -\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}[\tilde{g}(s+1)\cos\theta s + \mu_1 D(s)\cos(\pi/2+\theta)s]\frac{r^{-s}\,\mathrm{d}s}{s\mu_0\sin(\pi s/2)}. \tag{23}$$

Taking into account the considerations at the beginning of the third section, we take that $\varepsilon\in(0,\omega)$ in case $\alpha\in(0,1)$, while $\varepsilon\in(-\omega,0)$ for $\alpha\in(1,\infty)$.

We consider the two cases separately. When $0 < \alpha < 1$, using relations (19) and defining the angle ϕ calculated against the crack line ahead as $\phi = \pi/2 - \theta$, we obtain:

$$u_0(r,\phi) = C_0 - \frac{K_{\mathrm{III}}^{(1)}}{\mu_0(1-\alpha)} r^{1-\alpha} \operatorname{tg}\frac{\pi\alpha}{2} \cos(1-\alpha)(\pi-\phi) + O(r^{\min\{1,2(1-\alpha)\}}), \quad r \to 0,$$

$$u_1(r,\phi) = \frac{K_{\mathrm{III}}^{(1)}}{\mu_1(1-\alpha)} r^{1-\alpha} \sin(1-\alpha)\phi + O(r^{\min\{1,2(1-\alpha)\}}), \quad r \to 0, \quad \phi \in (0,\pi/2),$$

$$u(r,\phi) = O(r^{-\omega}), \quad r \to \infty, \quad \phi \in (0,\pi), \tag{24}$$

where $\phi \in (\pi/2,\pi)$ in the first relation; the displacement discontinuity near crack tip is $C_0 = -\mu_1 \tau a_0^-$, but constant $K_{\mathrm{III}}^{(1)}$ is of the form: $K_{\mathrm{III}}^{(1)} = -\mu_1 a_0^- [\sin(\pi\alpha/2)]^{-1}$.

In the case for $\alpha > 1$ we find that

$$u_0(r,\phi) = \frac{K_{\mathrm{III}}^{(2)}}{\mu_1 \omega} \sqrt{\frac{\mu_1}{\mu_0}} r^{\omega} \cos\omega(\pi-\phi) + O(r^{\min\{2-\omega,\omega+\alpha-1\}}), \quad r \to 0, \quad \phi \in (\pi/2,\pi),$$

$$u_1(r,\phi) = \frac{K_{\mathrm{III}}^{(2)}}{\mu_1 \omega} r^{\omega} \sin\omega\phi + O(r^{\min\{2-\omega,\omega+\alpha-1\}}), \quad r \to 0, \quad \phi \in (0,\pi/2),$$

$$u(r,\phi) = \mathrm{const} + O(r^{1-\alpha}), \quad r \to \infty, \quad \phi \in (0,\pi), \tag{25}$$

where $K_{\mathrm{III}}^{(2)} = -\mu_1 a_0^+ [\cos(\pi\omega/2)]^{-1}$. The second terms of asymptotics in these relations is obtained from discussion of eqn (17).

Let us note that generalized stress intensity factors $K_{\mathrm{III}}^{(1)}$, $K_{\mathrm{III}}^{(2)}$ are the constants in the main terms of asymptotics of stresses near the crack tip and have dimension [N/m^2]. They coincide with the stress intensity factors (SIF) when the singularity exponent is equal to 0.5. Taking this fact into account, further on we shall call $K_{\mathrm{III}}^{(1)}$, $K_{\mathrm{III}}^{(2)}$ as SIF also.

In Fig. 3, a graph of the main exponent of stress singularity for the crack terminating at the bimaterial interface is presented with respect to the value of parameter α. Besides, a scheme demonstrating the distribution of a number of singular terms in the stress asymptotics in the neighbourhood of the crack tip is shown.

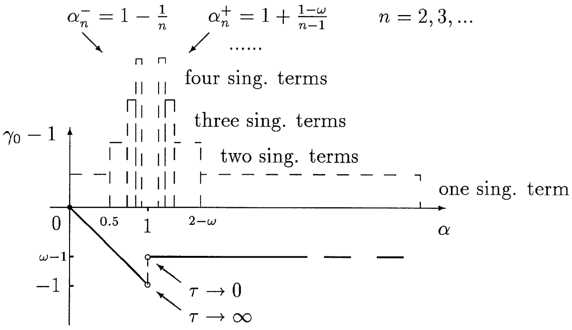

Fig. 3. Graph of the exponent of the main term of the stress asymptotics. Scheme of distribution of a number of singular terms in the asymptotics.

—For $\alpha = 0$, an exponential singularity of stresses near the crack tip does not exist for any values of the mechanical parameters μ_0, μ_1, τ. In this case, stress singularity appears only in domain Ω_1, and it has a logarithmic character (see Mishuris, 1997). Then there is displacement discontinuity along the bimaterial interfacial contact near the crack tip.

—If $\alpha \in (0, 0.5]$, only one singular term in asymptotics of stresses in the neighbourhood of the crack tip appears. Corresponding exponent in the interval $(-1, 0)$ is $\gamma_0 - 1 = -\alpha$, and does not depend on the values of the parameters μ_0, μ_1, τ.

—For case $\alpha \in (0.5, 1)$, or more precisely for $\alpha \in (\alpha_n^-, \alpha_{n+1}^-)$, $(n = 2, 3, \ldots)$, where $\alpha_n^- = 1 - 1/n$, there are exactly n singular terms in the asymptotics of stresses with the exponents $\gamma_0 - 1 = -\alpha$, $\gamma_j^- - 1 = (j+1)(1-\alpha) - 1 \in (-1, 0)$, $j = 1, \ldots, n-1$ (see diagram in Fig. 3). The number of n is calculated by the relation $n = [(1-\alpha)^{-1}]$, where by $[x]$ we denote the integral part of the real value x. Let us note that $n \to \infty$ when $\alpha \to 1$. Constants in the corresponding terms of poles of the function $D(s)$ are calculated as follows:

$$a_j^- = \lim_{s \to -\gamma_j^-} (s + \gamma_j^-) D(s) = \frac{\Delta(\gamma_j^-)}{\mu_1 \tau \gamma_j^-} a_{j-1}^-, \quad j = 1, \ldots, n. \tag{26}$$

In the last two cases ($\alpha \in (0, 1)$), the displacement discontinuity near the crack tip also appears.

—If $\alpha = 1$ there is just one singular term of asymptotics with the exponent $\gamma_0 - 1 \in (-1, \omega - 1)$, depending essentially on the values of the mechanical parameters μ_0, μ_1, τ. Corresponding equation to calculate this singularity and graphs of the exponent, as a function of the parameters, are presented in a paper by Mishuris (1997). For example, when $\tau \to 0$ we have $\gamma_0 - 1 \to \omega - 1$, this coincides with the result for the "ideal" contact. In this case ($\alpha = 1$) and the next one ($\alpha > 1$) the displacement field is continuous near the crack tip. [However, it is discontinuous on any distance from the crack tip along the bimaterial contact in view of conditions (8).]

—For case $\alpha \in (1, 2-\omega)$, or more precisely for $\alpha \in (\alpha_{n+1}^+, \alpha_n^+)$, $(n = 2, \ldots)$, where $\alpha_n^+ = 1 + (1-\omega)/(n+1)$, there are precisely n singular terms in asymptotics of stresses with the exponents $\gamma_0 - 1 = \omega - 1$, $\gamma_j^+ - 1 = j(\alpha - 1) + \omega - 1 \in (-1, 0)$, $j = 1, \ldots, n-1$ (see diagram in Fig. 3). As above $n \to \infty$ when $\alpha \to 1$, and $n = [(1-\omega)/(\alpha - 1) + 1]$. The corresponding constants as in (26) are calculated from the formulae:

$$a_j^+ = \lim_{s \to -\gamma_j^+} (s + \gamma_j^+) D(s) = \frac{\mu_1 \tau \gamma_j^+}{\Delta(\gamma_j^+)} a_{j-1}^+, \quad j = 1, \ldots, n. \tag{27}$$

—Finally, in case $\alpha \in [2-\omega, \infty)$, one singular term of the stress asymptotics appears. The corresponding exponent is of the form $\gamma_0 - 1 = \omega - 1$, and does not depend on the remaining parameters of the problem.

What is interesting to note is that there are two cases when the values of SIF $K_{\mathrm{III}}^{(1)}$, $K_{\mathrm{III}}^{(2)}$ in the main terms of asymptotics (24), (25) can be calculated in a closed form. Namely, if $\alpha = 1/2$ and $\mu_1/\mu_0 = 1$, then the corresponding integral eqn (A5) degenerates (the kernel of the integral operator is equal to zero), and we obtain:

$$K_{\mathrm{III}}^{(1)} = \frac{-2\sqrt{2}}{\pi \tau \mu_0} \left\{ \mu_1 \pi \int_0^\infty t^{-1/2} h(t) \check{G}(t)\, \mathrm{d}t - 2\Gamma'(1) \tilde{g}(1) + \tilde{g}'(1) \right\}. \tag{28}$$

In this case, we have the usual square root stress singularity near the crack tip, but the dependence of $K_{\rm III}^{(1)}$ on the parameter τ has a nonlinear character. [As it follows from (A5), the functions $h(t)$, $\breve{G}(t)$ also depend on parameter τ.]

The second case is when $\alpha = 1+\omega$, but the other parameters μ_1/μ_0, τ can be of arbitrary values. Then the exponent of the stress singularity is equal to $\omega-1$, and the corresponding coefficient $K_{\rm III}^{(2)}$ depends in a linear way on the parameter τ, and is calculated from the relation (22) taking into account (16):

$$K_{\rm III}^{(2)} = \frac{\mu_1\omega\mu_0}{\pi(\mu_0+\mu_1)\cos(\pi\omega/2)}\left\{\tau\tilde{g}(1)+\frac{\tilde{g}(1-\omega)}{\mu_0\omega\sin(\pi\omega/2)}\right\}. \tag{29}$$

6. Numerical results and discussion

Now we present numerical results concerning the stress intensity factors for loading $g(r) = p\delta(r-1)$, where constant p has dimension [N/m^2] ($\delta(r-1)$ the Dirac delta applied on the unit distance from the crack tip). Different values of the remaining dimensionless mechanical parameters of the problem μ_1/μ_0, α, $\tau_0 = \mu_0\tau$ are considered. Further on, we shall consider both the cases $\alpha\in(0, 0.5]$ and $\alpha\in[2, \infty)$ separately, since they have their specific features (there is always only one singular term of stresses near the crack tip). Numerical results for the cases $\alpha = 0$ and $\alpha = 1$ are presented in a paper by Mishuris (1997).

The case $\alpha\in(0, 0.5)$

First of all we investigate the influence of normalized parameter $\tau_0 = \tau\mu_0$ on the coefficients in (24). Thus, in Fig. 4, diagrams of the normalized SIF $K_{\rm III}^{(1)}/p$ (Fig. 4a) and the normalized jump of displacement near crack tip $C_0\mu_0/p$ (Fig. 4b) are presented in a logarithmic scale as functions of τ_0 for different values of ratio μ_1/μ_0 and for $\alpha = 0.1$. It is evident that the corresponding curves are straight lines for all values of the parameter τ_0 under consideration. Consequently, we can conclude that

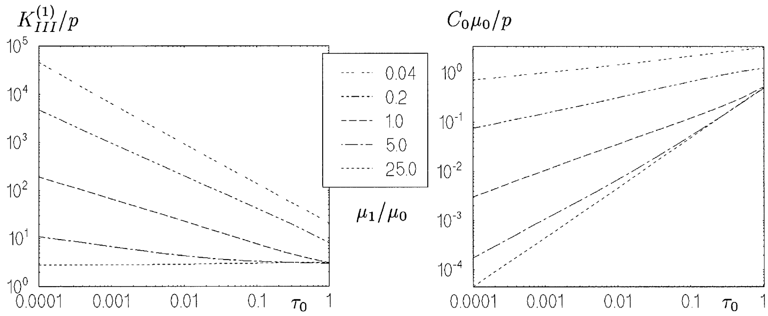

Fig. 4. Graphs of the normalized SIF $K_{\rm III}^{(1)}/p$ and the normalized jump of displacement $C_0\mu_0/p$ [see (24)] in the logarithmic scale as functions of the normalized parameter $\tau_0 = \tau\mu_0$ in the case $\alpha = 0.1$ for different values of ratio μ_1/μ_0, and loading $g(r) = p\delta(r-1)$ [N/m^2].

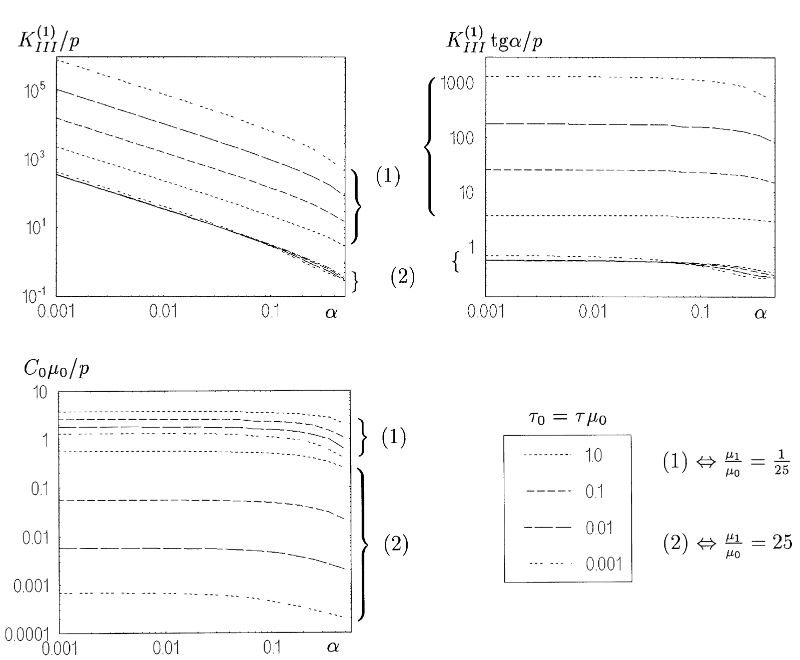

Fig. 5. Graphs of the coefficients of asymptotics (24) as functions of the exponent $\alpha \in (0, 0.5)$ in logarithmic scale for different magnitudes of the normed parameter $\tau_0 = \tau\mu_0$ and for two values of parameter $\mu_1/\mu_0 = 0.04$; 25 [the corresponding curves are denoted by (1) and (2), respectively].

$$C_0 \sim \tau_0^{\omega_*}, \quad K_{\mathrm{III}}^{(1)} \sim \tau_0^{\omega_*-1}. \tag{30}$$

The value of ω_* calculated numerically is equal to $\omega_* = \omega + \alpha$ with an accuracy not exceeding 2%, as it is to be expected. Hence, if $\tau_0 \to 0$, then asymptotics (24) coincides with that for the "ideal" bimaterial contact ($\tau_0 = 0$). Moreover, relations (30) make it possible to calculate constants $C_0\mu_0$, $K_{\mathrm{III}}^{(1)}$ in (24) for all values of τ_0 using only a piece information of a single sufficiently small value of τ_0.

Influence of parameter α from interval (0, 0.5) on the coefficients of asymptotics (24) is illustrated in Fig. 5. Two cases of the ratio of the shear moduli are presented. By symbol (1), we denote the respective curves when the crack terminates on a soft half-plane ($\mu_1/\mu_0 = 1/25$), (2) however, corresponds to ($\mu_1/\mu_0 = 25$). As it follows from Fig. 5a, graphs for SIF $K_{\mathrm{III}}^{(1)}$ increase when $\alpha \to 0$. Constant $K_{\mathrm{III}}^{(1)}\,\mathrm{tg}(\pi\alpha/2)$ presented by the first term of the asymptotics of function u_0 does not increase. This fact coincides with the distribution of stresses near the crack tip arising in the limiting case $\alpha = 0$.

Finally, in Fig. 6, we present the normalized coefficients $C_0\mu_0/p$, $K_{\mathrm{III}}^{(1)}/p$ as functions of the shear moduli ratio μ_1/μ_0 for different values of τ_0 and $\alpha = 0.1$. Thus, for large values of the ratio μ_1/μ_0 (the crack terminates on a stronger half-plane), magnitudes of SIF $K_{\mathrm{III}}^{(1)}$ do not depend much on parameter τ_0. This fact coincides with the results from Fig. 5a.

On the other hand it is not true for $C_0\mu_0$. Namely, the values of displacement discontinuity near crack tip along the interface are almost equal for all values of the parameter τ_0 when $\mu_1/\mu_0 \to 0$ (the

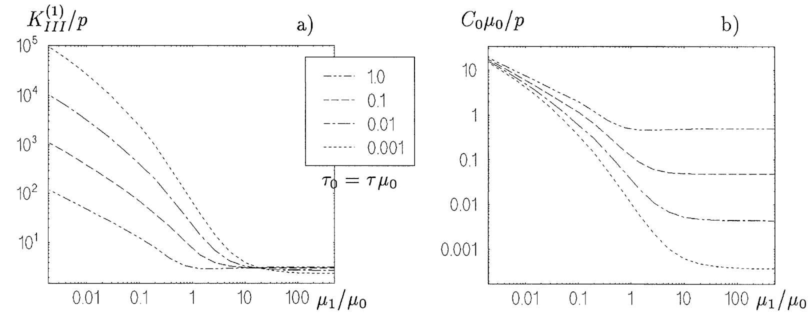

Fig. 6. Graphs of the normalized coefficients $K^{(1)}_{\mathrm{III}}/p$ and $C_0\mu_0/p$ as functions of ratio μ_1/μ_0 in logarithmic scale for parameter $\alpha = 0.1$ and different values of parameter $\tau_0 = \tau\mu_0$.

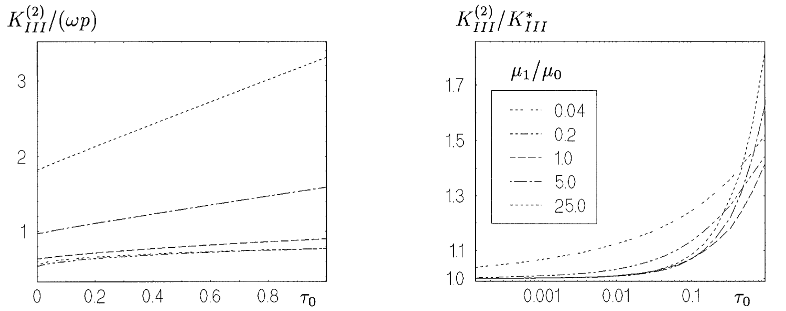

Fig. 7. Graphs of the normalized coefficients $K^{(2)}_{\mathrm{III}}/(\omega p)$ and $K^{(2)}_{\mathrm{III}}/K^{*}_{\mathrm{III}}$ of eqn (25) as functions of parameter $\tau_0 = \tau\mu_0$ in the case $\alpha = 2$ and different values of ratio μ_1/μ_0.

crack terminates on a soft half-plane). Analogous behaviour can be seen in Fig. 5c. Both the results are to be expected.

The case $\alpha \in (2-\omega, \infty)$

Let us remember that in this case the stress singularity exponent does not depend on all remaining parameters and is equal to $\omega - 1$ (see diagram in Fig. 3). Consequently, we can compare the respective unique value of $K^{(2)}_{\mathrm{III}}$ from (25) with the coefficient

$$K^{*}_{\mathrm{III}} = \frac{2\mu_1 \tilde{g}(1-\omega)}{\pi(\mu_0+\mu_1)\sin\pi\omega}, \tag{31}$$

corresponding to the "ideal" contact condition ($\tau_0 = 0$). Beside value $K^{(2)}_{\mathrm{III}}/K^{*}_{\mathrm{III}}$, we present also diagrams of ratio $K^{(2)}_{\mathrm{III}}/\omega$ arising while critical crack opening criteria (Dugdale, 1960; Wells, 1961; McClintock, 1958), or the effective stresses criteria (Novozhylov, 1969; Seweryn, 1994; Seweryn and Mroz, 1995) are applied.

In Fig. 7, diagrams of parameters $K^{(2)}_{\mathrm{III}}/(\omega p)$, $K^{(2)}_{\mathrm{III}}/K^{*}_{\mathrm{III}}$ are presented as functions of parameter τ_0 for different values of the shear moduli ratio μ_1/μ_0 and for $\alpha = 2.0$. When $\tau_0 > 0.05$, all curves in Fig. 7a become straight lines. Consequently there is a linear dependence of $K^{(2)}_{\mathrm{III}}$ on this parameter for such values of τ_0. Let us remember that in the case $\alpha = 1+\omega$ relation (29) for $K_{\rm III}^{(2)}$ behaves similarly for all values of τ_0. The next conclusion which can be drawn in that parameter τ_0 has little effect on the value of SIF for $\tau_0 < 0.1$. (The corresponding ratio $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ is less than 1.1.)

This fact is more evident in Fig. 8b, where parameter $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ is presented as a function of the shear moduli ratio μ_1/μ_0, for different values of τ_0 and for $\alpha = 2.0$. On the other hand, for $\mu_1/\mu_0 \to 0$ SIF $K_{\rm III}^{(2)}$ tends to zero, and the stress singularity $\omega - 1$ tends to minus one ($\omega \to 0$). However, the combined coefficient $K_{\rm III}^{(2)}\,{\rm d}^{\omega}/\omega$ arising within the frames of any of the fracture criteria mentioned above does not lead to a paradoxical conclusion. Although the parameter $d \ll 1$ has different mechanical interpretation for each of the criterion, the coefficient $K_{\rm III}^{(2)}\,{\rm d}^{\omega}/\omega$ decreases monotonically as μ_1/μ_0 increases.

Now we investigate the influence of parameter α on SIF. In Fig. 9, diagrams of $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ act as functions of parameter τ for two values of the shear moduli ratio μ_1/μ_0 ($\mu_1/\mu_0 = 50$—Fig. 9a; $\mu_1/\mu_0 = 1/50$—Fig. 9b) and various values of α. In Fig. 10, the values of $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ are presented as functions of α for various magnitudes of τ_0 in the case of: $\mu_1/\mu_0 = 25$ in Fig. 10a; and $\mu_1/\mu_0 = 1/25$ in Fig. 10b. As it can be easily seen, parameter α has the greatest influence on $K_{\rm III}^{(2)}$ when the crack terminates on a soft half-plane (for small values of the ratio μ_1/μ_0). Besides, in this case, $K_{\rm III}^{(2)}$ depends on τ_0 monotonically. In the second case, when the crack is situated in front of a stronger material, the behaviour of $K_{\rm III}^{(2)}$ against parameter α is more complicated.

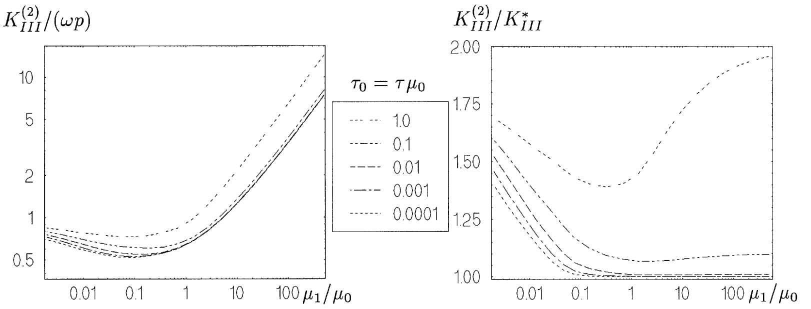

Fig. 8. Graphs of the normalized coefficients $K_{\rm III}^{(2)}/(\omega p)$ and $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ of asymptotics (25) as functions of ratio μ_1/μ_0 in the case $\alpha = 2$ and different values of ratio $\tau_1 = \tau\mu_0$.

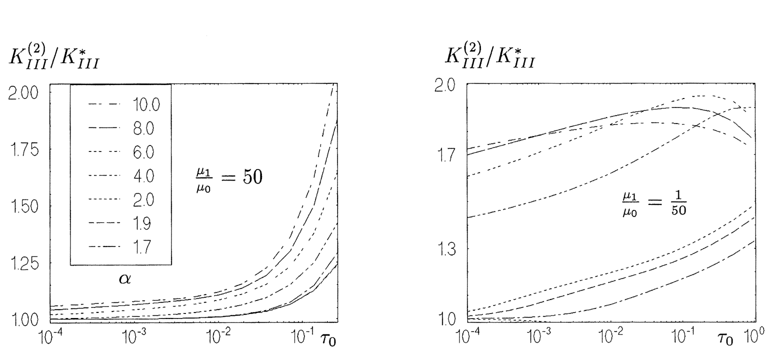

Fig. 9. Graphs of the normalized coefficient $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ as functions of parameter $\tau_0 = \tau\mu_0$ for different values of α for two values of ratio $\mu_1/\mu_0 = 50$; 0.02.

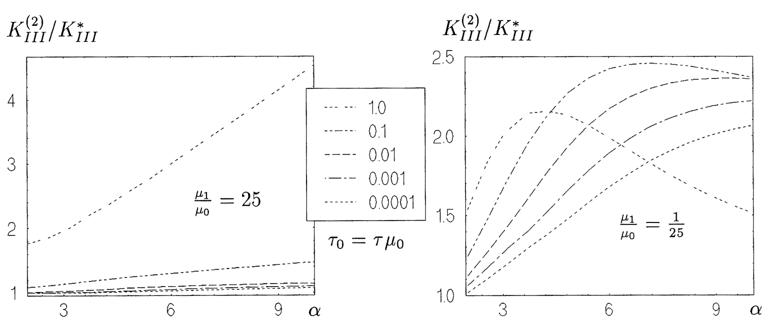

Fig. 10. Graphs of the normalized coefficient $K_{\rm III}^{(2)}/K_{\rm III}^{*}$ as functions of parameter α for different values of τ_0 for two values of ratio $\mu_1/\mu_0 = 25$; 0.04.

It seems to be natural to expect that $K_{\rm III}^{(2)}$ is closer to $K_{\rm III}^{*}$ as $\alpha \to \infty$. However, this is not the case in our considerations. This paradox can be explained easily. Namely, in the model under consideration, it has been assumed that condition $(8)_1$ is satisfied along the whole bimaterial contact. However, when $r \to \infty$ and $\alpha > 1$ (Fig. 2e), the relative thickness of the intermediate zone is not small, this is in contradiction to the assumption at the beginning of the paper. In the result, the displacement does not vanish at infinity [see (25)]. To eliminate such an inconsistency, it is necessary to correct the respective interfacial condition as follows:

$$[u] = f(r,\tau,\alpha)\sigma_n, \quad f(r,\tau,\alpha) = \begin{cases} \tau r^{\alpha}, & 0 < r \leqslant 1, \\ \tau, & 1 < r < \infty. \end{cases}$$

In this paper we shall not investigate the corresponding boundary value problem.

In regard to the cases where the parameter $\alpha \in (0.5, 2)$, the coefficient $K_{\rm III}$ of the first asymptotic term cannot be used in fracture mechanics analysis as a unique characteristic. Coefficients in the next singular terms should be taken into account in the analysis. Let us note in this connection that there exists a recurrent relation between the sequential coefficients $K_{\rm III}(j)$ and $K_{\rm III}(j+1)$ which follows from the respective relationships (26) and (27).

7. Conclusions

As it would be expected, geometry of the thin intermediate zone between the different materials influences essentially the stress singularity near the crack tip terminating at the interface.

Whereas, the exponent of the stress singularity for the "ideal" model of the bimaterial interface (Fig. 2a) is equal to $\omega - 1$ and there is only one singular term of asymptotics of stress near the crack tip, situation varies for non-ideal contact conditions. Graph of the main exponent of stress singularity as a function of parameter α determining the geometry of the intermediate zone is presented in Fig. 3. Let us remember that in the vicinity of point $\alpha = 1$ (gap on the curve) asymptotic expansion of stress contains a number n of singular terms, and n tends to infinity as $\alpha \to 0$. Consequently, magnitudes of stresses and displacements some distance from the crack tip change continuously against α.

Influence of all mechanical parameters of the problem on the value of SIF are illustrated and discussed in the previous section and for the cases $\alpha = 0$, $\alpha = 1$ in a paper by Mishuris (1997).

Of course, these theoretical results should be experimentally verified. Let us note in this connection that for a practical intermediate zone the value of parameter τ_0 is very small, as a rule. Consequently, on the distance from the crack tip $r \sim \tau_0$, the asymptotics of stress coincides with that for the "ideal" bimaterial contact. This fact should be taken into account when the corresponding experimental results will be interpreted.

Although in most of the considered cases the stress singularity is not equal to -0.5, in general, fracture mechanics analysis can be done by the critical crack opening criteria or the effective stress criteria mentioned above. Corresponding investigation is not a goal of this paper.

Acknowledgements

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Appendix

Consider eqn (17) in the case $\alpha \in (0,1)$. To solve the problem we introduce a new function $F(s)$ by the relation:

$$D(s) = \Gamma\left(1+\frac{s}{1-\alpha}\right)\left[F(s)-\frac{2(1-\alpha)}{\mu_1 \pi s}\tilde{g}(1)\right]\sin\frac{\pi s}{2(1-\alpha)}, \tag{A1}$$

which is analytic in a strip $-\beta < \Re s < \min\{\omega, 2(1-\alpha)\}$ with some value $\beta > 1-\alpha$ in view to (18). Note, that $D(0) = -\mu_1^{-1}\tilde{g}(1)$ as it should be in (16). Substitute (A1) in (17) we obtain the corresponding functional equation:

$$F(s)+\mu_1\tau(1-\alpha)\Delta^{-1}(s)\,\mathrm{ctg}\frac{\pi s}{2(1-\alpha)}F(s+\alpha-1) = G(s), \tag{A2}$$

where the function

$$G(s) = -\left\{\mu_0\Delta(s)\Gamma\left(1+\frac{s}{1-\alpha}\right)\sin\frac{\pi s}{2(1-\alpha)}\sin\frac{\pi s}{2}\right\}^{-1}\tilde{g}(s+1)$$
$$+\frac{2(1-\alpha)}{\pi\mu_1 s}\tilde{g}(1)+\frac{2\tau(1-\alpha)^2}{\pi\Delta(s)(s+\alpha-1)}\mathrm{ctg}\frac{\pi s}{2(1-\alpha)}\tilde{g}(1),$$

is analytic in the strip $|\Re s| \leqslant \min\{\omega, 2|1-\alpha|\}$, and tends to zero when $|\Im s| \to \infty$ so that $G(s) = O(s^{-1})$. It is more convenient for us to rewrite eqn (A1) in an equivalent form:

$$F(s)+\mu_1\tau|1-\alpha|\Delta^{-1}(s)\operatorname{ctg}\frac{\pi s}{2|1-\alpha|}F(s+\alpha-1) = G(s). \tag{A3}$$

Let us note here, that

$$\Delta^{-1}(s)\operatorname{ctg}\frac{\pi s}{2|1-\alpha|} = \frac{\mu_0}{\mu_0+\mu_1}+\Psi(s),$$

$$\Psi(s) = \frac{\mu_0}{\mu_0+\mu_1}\left\{\sin\left[\pi s\left(1-\frac{1}{2|1-\alpha|}\right)\right]+\kappa\sin\frac{\pi s}{2|1-\alpha|}\right\}\left\{(\cos\pi s-\kappa)\sin\frac{\pi s}{2|1-\alpha|}\right\}^{-1},$$

where $\Psi(s)+\mu_0/(\mu_0+\mu_1)$ is an even analytic function in the strip $|\Re s| \leqslant \min\{\omega, 2|1-\alpha|\}$, which is not equal to zero along the imaginary axis. Besides, there is an estimation: $\Psi(s) = O(\exp(-\pi \min\{1, |1-\alpha|^{-1}\}|\Im s|)$, when $|\Im s| \to \infty$.

We shall find the solution of the functional eqn (A2) in the form of the Mellin transform:

$$F(s) = \int_0^\infty f(t)t^{s-\alpha}\,\mathrm{d}t, \quad F(s+\alpha-1) = \int_0^\infty f(t)t^{s-1}\,\mathrm{d}t, \tag{A4}$$

where $-\beta < \Re s < \min\{\omega, 2|1-\alpha|\}$. Then function $f(t)$ should belong to space $f\in L^{1,\delta_1\delta_2}(\mathbb{R}_+)$, $\delta_1 > -\beta+1-\alpha$, $\delta_2 < \min\{1+\omega-\alpha, 3(1-\alpha)\}$. Here $L^{p,\delta_1,\delta_2}(\mathbb{R}_+)$ is Banach space of summable functions of the norm (see Mishuris and Olesiak, 1995)

$$\|u\|^{p,\delta_1,\delta_2} = \left(\int_0^\infty |u(\xi)|^p\rho^p_{\delta_1,\delta_2}(\xi)\frac{\mathrm{d}\xi}{\xi}\right)^{1/p}, \quad \rho_{\delta_1,\delta_2}(\xi) = \begin{cases}\xi^{\delta_1}, & 0<\xi<1;\\ \xi^{\delta_2}, & 1<\xi<\infty.\end{cases}$$

Substituting (A4) in (A2) and using the inverse Mellin transform, we obtain the singular integral equation:

$$f(t)+\mu_1\tau|1-\alpha|\int_0^\infty \check{\Psi}(t/\xi)h(t)f(\xi)\,\mathrm{d}\xi/\xi = h(t)\check{G}(t), \tag{A5}$$

where functions $\check{G}(t)$, $\check{\Psi}(t)$, $h(t)$ are defined as follows:

$$\check{G}(t),\check{\Psi}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} G(s), \quad \Psi(s)t^{-s}\,\mathrm{d}s, \quad h(t) = \left[t^{1-\alpha}+\frac{\mu_0\mu_1\tau|1-\alpha|}{\mu_0+\mu_1}\right]^{-1}.$$

Here $h\check{G}\in L^{p,\delta_1,\delta_2}(\mathbb{R}_+)$ for any $\delta_1 > -\min\{\omega, 2(1-\alpha)\}$, $\delta_2 < \min\{\omega+1-\alpha, 3(1-\alpha)\}$, then basing this on the results from a paper by Mishuris and Olesiak (1995) and the properties of functions $\check{\Psi}$, h, we can formulate the following:

Theorem 1. Let $\alpha\in[0,1)$, $p\in[1,\infty)$, $-\min\{\omega, 2(1-\alpha)\} < \delta_1 < \delta_2 < \min\{1+\omega-\alpha, 3(1-\alpha)\}$, then eqn (A5) has a unique solution in the space $L^{p,\delta_1,\delta_2}(\mathbb{R}_+)$, which can be calculated by projectional methods (see Gohberg and Feldman, 1971). Consequently, all Mellin integrals in relation (20) have a sense.

The case $\alpha > 1$ is considered in a similar way. All relations and the singular integral eqn (A5) still hold true, and we should only take into account the strips of analyticity of the functions. Namely, function $F(s)$ in (A1) is analytic in the strip $-\min\{\omega, 2(\alpha-1)\} < \Re s < \beta$, for some $\beta > \alpha-1$, function $G(s)$ is analytic in the strip $|\Re s| \leqslant \min\{\omega, 2(\alpha-1)\}$, but the behaviour of function $h(t)$ at zero and infinity will be different in comparison with the case for $0 < \alpha < 1$.

Theorem 2. Let $\alpha \in (1, \infty), p \in [1, \infty)$, $-\min\{\omega+\alpha-1, 3(\alpha-1)\} < \delta_1 < \delta_2 < \min\{\omega, 2(\alpha-1)\}$, then eqn (A5) has a unique solution in the space $L^{p,\delta_1,\delta_2}(\mathbb{R}_+)$, which can be calculated by projectional methods.

References

Atkinson, C. (1977) On stress singularities and interfaces in linear elastic fracture mechanics. *International Journal of Fracture* **13**(6), 807–820.

Atkinson, C. and Javaherian, H. (1980) Evaluation of stress intensity factors associated with bimaterial cracks. *J. Inst. Maths Applics* **26,** 235–258.

Cherepanov, G. P. (1979) *Mechanics of Brittle Fracture*. McGraw-Hill, New York.

Comninou, M. (1979) An overview of interface crack. *Engineering Fracture Mechanics* **37,** 197–208.

Dugdale, D. S. (1960) Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids* **8**(2), 100–108.

Erdogan, F., Kaya, A. C. and Joseph, P. F. (1991) The mode III crack problem in bonded materials with a non-homogeneous interfacial zone. *Journal of Applied Mechanics* **58,** 419–427.

Gohberg, I. C. and Feldman, N. A. (1971) *Equations in Convolutions and Projectional Methods of their Solutions*. Nauka, Moscow (in Russian).

He, M. Y. and Hutchinson, J. W. (1989a) Kinking of a crack out of an interface. *Journal of Applied Mechanics* **56,** 270–278.

He, M. Y. and Hutchinson, J. W. (1989b) Crack deflection at an interface between dissimilar elastic materials. *International Journal of Solids and Structures* **25,** 1053–1067.

McClintock, F. A. (1958) Ductile fracture instability in shear. *Journal of Applied Mechanics* **25**(4), 582–587.

Mishuris, G. S. (1985) On models of the interface between two elastic media one of which weakened by an angular cut. *Vestnik Leningradskogo Un-ta*, **22,** 62–66 (in Russian, translated into English).

Mishuris, G. S. (1986) A plane problem of elasticity theory for a layered medium with semi-infinity crack perpendicular to the boundary between the layers. *Studies in Elasticity and Plasticity*, Leningrad University Publ., **15,** 82–96 (in Russian, translated into English).

Mishuris, G. S. (1997) Influence of interfacial models on a stress field near a crack terminating at a bimaterial interface. *International Journal of Solids and Structures* **34**(1), 31–46.

Mishuris, G. S. and Olesiak, Z. S. (1995) On boundary problems in fracture of elastic composites. *European Journal of Applied Mathematics* **6,** 591–610.

Novozhilov, V. V. (1969) On necessary and sufficient criterion of brittle fracture. *Applied Math. Mech.* **33**(5), 797–812 (in Russian, translated into English).

Rice, J. R. (1988) Elastic fracture mechanics concepts for interfacial cracks. *Journal of Applied Mechanics* **55,** 98–103.

Seweryn, A. (1994) Brittle fracture criterion for structures with sharp notes. *Engineering Fracture Mechanics* **47,** 673–681.

Seweryn, A. and Mróz, Z. (1995) A non-local stress failure condition for structural elements under multiaxial loading. *Engineering Fracture Mechanics* **51,** 955–973.

Wells, A. A. (1961) Critical tip opening as fracture criterion. In: *Proceedings of the Crack Propagation Symposium Grangield, Cranfield*, **1,** 210–221.

Zak, A. R. and Williams, M. L. (1963) Crack point stress singularities at a bimaterial interface. *Journal of Applied Mechanics* **30**(1), 142–143.